

# 物理で使う数学 / Jacobian

# Jacobian

独立変数 $(x, y)$ に関する2つの関数,

$$\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$$

に関するJacobianは,

$$J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial v}{\partial x}\right)_y & \left(\frac{\partial v}{\partial y}\right)_x \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial y}\right)_x - \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial x}\right)_y$$

のように定義される。独立変数が明らかなときは省略して $J(u, v)$ と書かれることもある

# Jacobianの性質

$$(1) \quad J(x, y) = -J(y, x) = 1$$

$$(2) \quad J(u, v) = -J(v, u)$$

$$(3) \quad J(u, v) = J(v, -u)$$

$$(4) \quad J(u, y) = \frac{\partial(u, y)}{\partial(x, y)} = \left( \frac{\partial u}{\partial x} \right)_y$$

$$(5) \quad J(x, v) = \frac{\partial(x, v)}{\partial(x, y)} = \left( \frac{\partial v}{\partial y} \right)_x$$

# Jacobian

$$(1) \quad J(x, y) = -J(y, x) = 1$$

$$J(x, y) = \frac{\partial(x, y)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} \end{vmatrix} = \left( \frac{\partial x}{\partial x} \right) \left( \frac{\partial y}{\partial y} \right) - \left( \frac{\partial x}{\partial y} \right) \left( \frac{\partial y}{\partial x} \right) = 1$$

$$J(y, x) = \frac{\partial(y, x)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial x}{\partial y} \end{vmatrix} = \left( \frac{\partial y}{\partial x} \right) \left( \frac{\partial x}{\partial y} \right) - \left( \frac{\partial y}{\partial y} \right) \left( \frac{\partial x}{\partial x} \right) = -1$$

$$\therefore \quad J(x, y) = -J(y, x) = 1$$

# Jacobian

$$(2) \quad J(u, v) = -J(v, u)$$

$$J(u, v) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial x} \right)$$

$$\begin{aligned} J(v, u) &= \frac{\partial(v, u)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{vmatrix} = \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) - \left( \frac{\partial v}{\partial y} \right) \left( \frac{\partial u}{\partial x} \right) \\ &= - \left\{ \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial x} \right) \right\} = -J(u, v) \end{aligned}$$

# Jacobian

$$(3) \quad J(u, v) = J(v, -u)$$

$$J(u, v) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial x} \right)$$

# Jacobian

$$(4) \quad J(u, y) = \frac{\partial(u, y)}{\partial(x, y)} = \left( \frac{\partial u}{\partial x} \right)_y$$

$$J(u, y) = \frac{\partial(u, y)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{vmatrix} = \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial y}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial y}{\partial x} \right) = \left( \frac{\partial u}{\partial x} \right)_y$$

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# Jacobian

$$(5) \quad J(x, v) = \frac{\partial(x, v)}{\partial(x, y)} = \left( \frac{\partial v}{\partial y} \right)_x$$

$$J(x, v) = \frac{\partial(x, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \underbrace{\left( \frac{\partial x}{\partial x} \right)}_{\parallel} \left( \frac{\partial v}{\partial y} \right) - \underbrace{\left( \frac{\partial v}{\partial y} \right)}_{\parallel} \left( \frac{\partial x}{\partial x} \right) = \left( \frac{\partial v}{\partial y} \right)_x$$

# Maxwell relation

$$\left( \frac{\partial}{\partial V} \left( \frac{\partial U}{\partial S} \right)_V \right)_S = \left( \frac{\partial}{\partial S} \left( \frac{\partial U}{\partial V} \right)_S \right)_V \Rightarrow \therefore \left( \frac{\partial T}{\partial V} \right)_S = - \left( \frac{\partial p}{\partial S} \right)_V$$

$$-\left( \frac{\partial}{\partial V} \left( \frac{\partial F}{\partial T} \right)_V \right)_T = - \left( \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial V} \right)_T \right)_V \Rightarrow \therefore \left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial p}{\partial T} \right)_V$$

$$\left( \frac{\partial}{\partial p} \left( \frac{\partial G}{\partial T} \right)_p \right)_T = \left( \frac{\partial}{\partial T} \left( \frac{\partial G}{\partial p} \right)_T \right)_p \Rightarrow \therefore - \left( \frac{\partial S}{\partial p} \right)_T = \left( \frac{\partial V}{\partial T} \right)_p$$

$$\left( \frac{\partial}{\partial p} \left( \frac{\partial H}{\partial S} \right)_p \right)_S = \left( \frac{\partial}{\partial S} \left( \frac{\partial H}{\partial p} \right)_S \right)_p \Rightarrow \therefore \left( \frac{\partial T}{\partial p} \right)_S = \left( \frac{\partial V}{\partial S} \right)_p$$

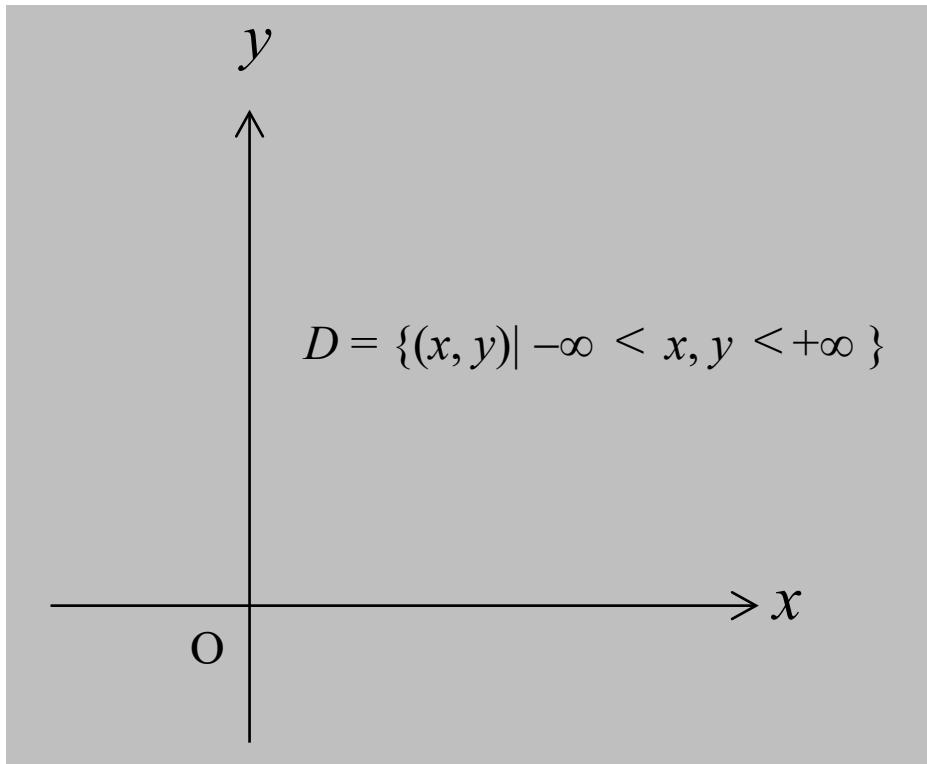
$$\left. \begin{array}{l} \frac{\partial(T,S)}{\partial(p,V)} = 1 \end{array} \right\}$$

# 参考 : Gauss積分

$$I = \int_{-\infty}^{+\infty} e^{-ax^2} dx \text{ を求めよ。}$$

ポイント :  $I^2$ を計算する。

$$I^2 = \left( \int_{-\infty}^{+\infty} e^{-ax^2} dx \right)^2 = \left( \int_{-\infty}^{+\infty} e^{-ax^2} dx \right) \left( \int_{-\infty}^{+\infty} e^{-ay^2} dy \right)$$



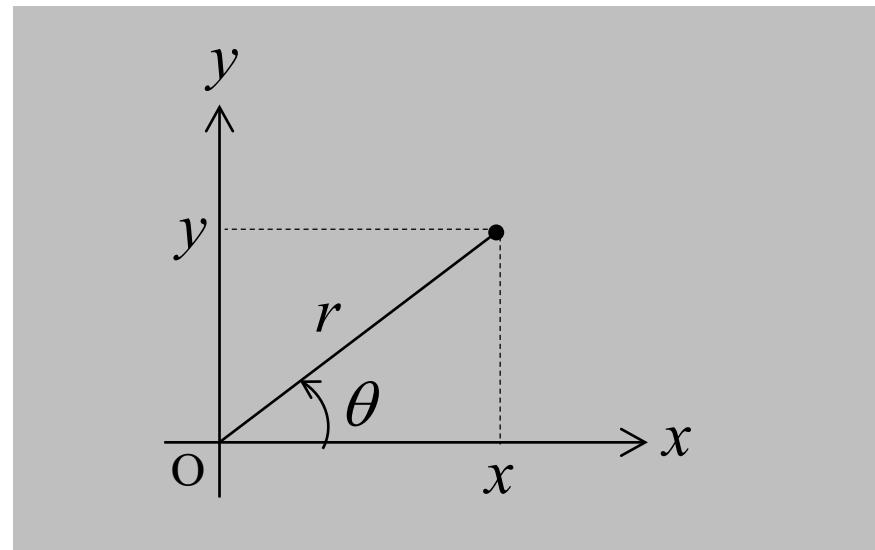
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-ax^2} e^{-ay^2} dx dy = \underbrace{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-a(x^2+y^2)} dx dy}_{\text{e}^{-a(x^2+y^2)} \text{ という関数を } xy \text{ 平面上の全領域で積分}}$$



$e^{-a(x^2+y^2)}$  という関数を  $xy$  平面上の全領域で積分

2次元の極座標を使うと,

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$



直交座標での積分範囲は

$$D = \{(x, y) | -\infty < x, y < +\infty\}$$

極座標での積分範囲は

$$D = \{(r, \theta) | r \geq 0, 0 \leq \theta \leq 2\pi\}$$

直交座標から極座標への変換で,

$$dxdy = J \left( \frac{x, y}{r, \theta} \right) dr d\theta \quad \text{← 重要}$$

ここで,

$$J \left( \frac{x, y}{r, \theta} \right) = \frac{\partial(x, y)}{\partial(r, \theta)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r \cos^2 \theta - (-r \sin^2 \theta) = r$$

$$\therefore dxdy = r dr d\theta \quad \text{← 重要}$$

$$D = \{(x, y) | -\infty < x, y < +\infty\}$$

$$= \{(r, \theta) | r \geq 0, 0 \leq \theta \leq 2\pi\}$$

$$I^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-a(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^{\infty} e^{-ar^2} r dr d\theta$$

$$= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\infty} e^{-ar^2} r dr \right) = 2\pi \left( \int_0^{\infty} e^{-ar^2} r dr \right)$$

ところで、

$$\frac{d}{dr}(e^{-ar^2}) = -2ar \times e^{-ar^2} \Rightarrow \therefore -\frac{d}{dr}\left(\frac{1}{2a}e^{-ar^2}\right) = e^{-ar^2} r$$

したがって、

$$\int_0^{\infty} e^{-ar^2} r dr = \left[ -\frac{1}{2a} e^{-ar^2} \right]_0^{\infty} = 0 - \left( -\frac{1}{2a} \right) = \frac{1}{2a}$$

$$I^2 = 2\pi \left( \int_0^{\infty} e^{-ar^2} r dr \right) = 2\pi \times \frac{1}{2a} = \frac{\pi}{a} \quad \therefore I = \int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

# 正規分布関数

$$f(X) = \frac{1}{\sqrt{2\pi}s} e^{-\frac{(X-\bar{X})^2}{2s^2}}$$

$$\int_{-\infty}^{+\infty} f(X)dX = \frac{1}{\sqrt{2\pi}s} \int_{-\infty}^{+\infty} e^{-\frac{(X-\bar{X})^2}{2s^2}} dX$$

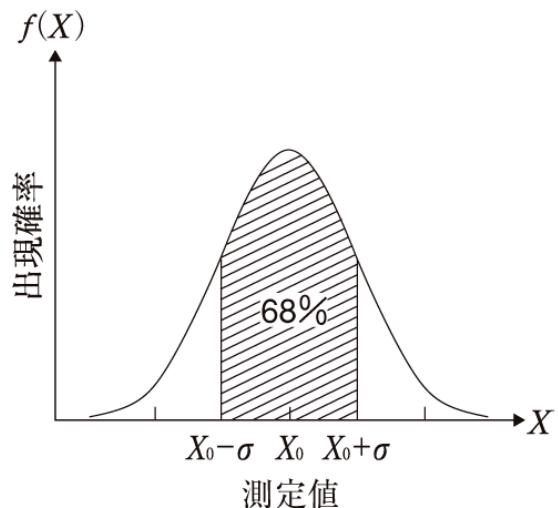


図 2.7 ガウス関数のグラフ

曲線の下の面積を 1 とする。斜線部の面積が 0.68 になる。斜線区間の端に曲線の変曲点（傾きが最大になる点）がある。

いま、変数変換として、

$$y = \frac{(X - \bar{X})}{\sqrt{2}s}$$

とすると、

$$\therefore \frac{dy}{dX} = \frac{1}{\sqrt{2}s} \Rightarrow \therefore dX = \sqrt{2}s dy$$

また、積分区間は

$$X : -\infty \rightarrow +\infty$$

$$y : -\infty \rightarrow +\infty$$

と対応するから

ガウス積分

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$\frac{1}{\sqrt{2\pi}s} \int_{-\infty}^{+\infty} e^{-\frac{(X-\bar{X})^2}{2s^2}} dX = \frac{1}{\sqrt{2\pi}s} \int_{-\infty}^{+\infty} e^{-y^2} \times \sqrt{2}s dy = \frac{1}{\sqrt{\pi}} \times \sqrt{\pi} = 1$$